Valuation of Mortgages by Using Lévy Models to Specify the State Variables for the Termination-Hazard and Recovery Rates

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Abstract:

In a mortgage valuation model, the early termination (i.e., prepayment and default) hazard rates and the recovery rate can be specified as multivariate affine functions that include the correlated stochastic state variables. For good capturing of the distributions for state variables, we specify that the state variables follow time-changed Lévy models. Accordingly, the early termination hazard rates and the recovery rate also follow time-changed Lévy processes. Three popular Lévy models, the normal, Variance Gamma (VG), and Negative Inverse Normal (NIG), were used to obtain the closed-form pricing formula for a mortgage and conduct numerical applications. Our empirical analyses reveal the following findings: VG model is better to fit the actual distributions of the interest rate and the change rate of the housing price than the normal and NIG models. Thus, mortgage valuation using a VG model should be better than that using the other two models. The mortgage value estimated by the normal model is the lowest among the three Lévy models, and the mortgage duration calculated by the normal model is also more variable than with the other two Lévy models. Our general pricing formula for a mortgage as described in this study can help market participants accurately value mortgages and effectively manage their risks.

Keywords: Valuation, Mortgage, Prepayment Risk, Default Risk, Lévy Process

1. Introduction

Outstanding U.S. bonds have grown from $1.93 trillion in 1980 to $42.68 trillion in 2018.[1] As shown in Figure 1, the market share of mortgage-related bonds has grown from 6% in 1980 to the highest level of 32% in 2007. The market share of mortgage-related bonds in bonds market was the largest during 1999-2010. Although this share has decreased since 2007, it was still the second largest segment of the U.S. fixed income markets from 2011 to 2018. It is obvious that mortgage-related securities play an important role in financial markets. Thus, an accurate and effective model for valuating mortgages is essential for market practitioners and financial researchers because it can help them undertake hedging analyses and allocate assets by considering the risks associated with mortgages and mortgage-related securities.

< Insert Figure 1 Here>

Whether a mortgage can be accurately valued depends mainly on accurate estimates of the early termination (i.e., prepayment and default) hazard rates and the recovery rate given default. Several studies have employed Cox's proportional hazard model, logistic regression, and Poisson regression to analyze the influential factors on the termination probabilities and the recovery rate (Cox and Oakes, 1984; Green and Shoven, 1986; Schwartz and Torous, 1989, 1993; Cunninghan and Capone, 1990; Quigley and Van Order, 1990, 1995; Smith, Sanchez and Lawrence, 1996; Hurt and Felsovalyi, 1998; Frye, 2000a, b; 2003; Lambrecht, Perraudin and Satchell, 2003; Dermine and de Carvalho, 2006, Liao et al., 2008; Tsai et al., 2009; Tsai and Chiang, 2012). Results from traditional researches indicate that the termination hazard rates (the recovery rate) are determined based on the basic hazard rates (the basic recovery rate) and some state variables, such as the interest rate and the change rate in the housing price (hereafter defined as the housing return rate). Accordingly, the

[1] Please refer to the SIFMA (Securities Industry and Financial Markets Association) Statistics website: http://www.sifma.org/research/statistics.aspx.

prepayment hazard rate, the default hazard rate and the recovery rate are usually specified as multivariate affine functions that include the correlated stochastic state variables (i.e., the interest rate and the housing return rate) (Duffee, 1999; Jarrow, 2001; Janosi et al., 2003; Capone, 2003; Liao et al., 2008).

On the valuation of a mortgage, traditional studies usually used geometric Brownian motion (BM) to specify the processes of state variables (Kau, Keenan and Smurov, 2004; Liao, Tsai, Chiang, 2008; Tsai, Liao, Chiang, 2009). However, empirical studies have challenged the assumption of a geometric BM process on the state variables. A number of researchers have demonstrated that the housing price exhibits a "heavy-tailed" distribution (McCulloch, 1986; Young and Graff, 1995; Graff et al., 1997, 1999; Young, 2007). Several studies have also provided empirical evidence of the jump magnitude phenomenon that indeed exits for housing prices (McCulloch, 1986; Young and Graff, 1995; Graff et al., 1997, 1999; Young, 2007). For interest rate, some authors have pointed out that the interest rate trajectories do not look like diffusion processes (Bjork, Kabanov and Runggaldier, 1997; Bjork, Masi, Kabanov and Runggaldier, 1997; Bjork and Christensen, 1999; Lekkos, 1999). Several studies provide empirical evidence that Lévy processes provide a better fit of bond returns or bond log-returns than those driven by BM (Raible, 2000; Hainaut and MacGilchrist, 2010). Accordingly, more and more researchers have incorporated the jump risks into their models to capture the effect of sudden changes in the house price when investigating issues regarding mortgages (Kau and Keenan, 1996; Chen et al., 2010; Chang et al., 2012; Tsai and Chiang, 2012; Calvo-Garrido et al., 2015). Also, interest rate models that admit jumps have been pursued in a number of studies, including Shirakawa (1991), Jarrow and Madan (1995), Bjork et al. (1997), Eberlien and Raible (1999), Raible (2000), Eberlein et al. (2005), Eberlein and Kluge (2006a,b, 2007), and Filipovic et al. (2010). In view of the above studies, although the assumption of geometric BM for state variables is analytically convenient, such assumptions no longer seem so realistic.

Recently, Lévy models have become increasingly popular in the discussion of the return process of financial assets. The empirical facts of excess kurtosis, skewness and fat tails can be modeled more realistically by a Lévy model (Heston 1993; Andersen and Lund, 1997; Pan, 2002; Eraker et al., 2003). Results from empirical studies have demonstrated that a Lévy model is superior to a normal model for assessing the goodness-of-fit of return distributions (Madan and Seneta, 1987; Barndorff-Nielsen, 1995; Raible, 2000; Seneta, 2004; Daal and Madan, 2005; Hainaut and MacGilchrist, 2010; Figueroa-Lopez et al., 2011). In addition, more and more researchers have adopted a Lévy model to optimally capture the distributions of termination probabilities and recovery rate (Jonsson et al., 2009; Fan et al., 2012; Maccaferri et al., 2013; Bo et al., 2014). Accordingly, this study describes a mortgage valuation model that includes the multivariate affine functions of termination hazard rates and the recovery rate with the correlated stochastic state variables following a time-changed Lévy process. To achieve this, the termination probabilities and the recovery rate can also be appropriately captured by a Lévy model, thus improving the accuracy of mortgage valuation.

A pure-jump Lévy process can display either finite activity or infinite activity. Intuitively speaking, a finite-activity jump process exhibits a finite number of jumps within any finite time interval, and an infinite-activity jump process generates an infinite number of jumps within any finite time interval. As is well known, there are many other members of the Lévy family that offer greater flexibility in modeling the asset price dynamic, including BM and the compound Poisson process (i.e., the jump-diffusion model) that capture a finite-activity jump process. In traditional studies, the analyses of mortgages usually use a Lévy model with the finite-activity jump process, such as the jump-diffusion model (Chen et al., 2010; Chang et al., 2012; Tsai and Chiang, 2012).

Previous studies have shown that a model with time changes, which allows for stochastically varying volatility, can capture the excess kurtosis and skewness of the underlying distribution (Heston 1993; Pan, 2002; Eraker et al., 2003). Moreover, an infinite-activity jump process can generate an infinite number of small and large movements within any finite time interval. This process can be captured by the Normal Inverse Gaussian (NIG) model of Barndorff-Nielsen (1997b), the Generalized Hyperbolic Class of Eberlein et al. (1998), the Variance Gamma (VG) Model of Madan and Milne (1991) and Madan et al. (1998 C13), the Carr-Geman-Madan-Yor (CGMY) Model of Carr et al. (2002) and the Finite Moment Log-stable (LS) Model of Carr and Wu (2003). Results from empirical studies have demonstrated that the distributions of logarithmic asset returns can often be fit extremely well by the NIG and the VG models (Madan et al., 1988; Barndorff-Nielsen, 1995; Rydberg, 1996, 1997a, 1999; Bu, 2007). These Lévy processes generally can be used to capture the time-changed appearance of return processes.

In this study, we incorporate a time-changed Lévy model with the infinite-activity jump process, such as a VG or NIG model, into our valuation model. We used the VG and NIG models to capture the interest rate and the housing return rate. In valuation model, we specified the termination hazard rates and the recovery rate as multivariate affine functions that include the correlated stochastic state variables (i.e., the interest rate and the housing return rate). Accordingly, the prepayment and default hazard rates, as well as the recovery rate, also follow VG and NIG models. Such specification should capture as many of their important style features as possible. To the best of our knowledge, our pricing model is the first to provide a pricing formula for mortgage valuation that incorporates the termination hazard rates and recovery rate following a time-changed Lévy model with an infinite-activity jump process.

We provide a numerical example for demonstrating how our valuation model can be used in practical applications. The maximum empirical likelihood method is used to estimate the parameters of the Lévy model (Qin and Lawless, 1994; Elgin, 2011). We then use these estimates and our pricing formula to price the mortgage. In previous studies, BM process usually has emerged as the benchmark process for describing asset return. Thus, to facilitate understanding of how accurately the different Lévy models price mortgages, we compare the results of letting the interest rate and the housing return rate be determined by the normal (BM), VG and NIG processes. Also, we provide the analyses of duration attributable to changes in the interest rate for these three Lévy models.

The remainder of this paper is structured as follows. Section 2 presents the general mortgage valuation framework. It describes the components of the mortgage contract and the definition of mortgage cash flow when the mortgage is active or terminated. It also explains the specification of the termination hazard rates and the recovery rate, and it illustrates the derivation of our mortgage valuation model. The specific implementation based on the three Lévy models (i.e., normal, VG and NIG) is discussed in Section 3. In Section 4, we examine which models best fit the actual underlying distributions of the housing return rate and interest rate. Moreover, this section describes the empirical methods for estimating the necessary parameters from historical data and presents the analyses of how the necessary parameters influence the mortgage value and duration. Also, we compare the pricing results for different termination rate and recovery rate specifications using these three Lévy models. The final section summarizes our findings.

2. The General Mortgage Valuation Model

This section presents a general mortgage valuation framework. Subsection 2.1 describes the components of the mortgage contract and the definition of mortgage payments, regardless of whether the mortgage is active, prepaid or in default. In Subsection 2.2, we specify the state variables, the termination hazard rates and recovery rates, and then we explain the derivation of the mortgage valuation formula.

2.1 The basic framework of mortgage valuation using a reduced-form model

In general, valuation models for a mortgage are constructed using either of two kinds of models: a structural-form model or a reduced-form model. The former type, pioneered by Dunn and McConnell (1981a,b), usually uses an American-style options model to value mortgages and examines the early termination risk (Kau, Keenan, Muller III and Epperson, 1993; Yang, Buist and Megbolugbe, 1998; Ambrose and Buttimer, 2000; Azevedo-Pereira, Newton and Paxson, 2003; Kau, Keenan and Smurov, 2004; Liao, Tsai, Chiang, 2008; Tsai, Liao, Chiang, 2009). The latter model assumes that the early termination events follow a Poisson distribution. The reduced-form model is extremely flexible because it can easily accommodate whatever explanatory variables are offered and whatever pattern of termination the data suggest. This model also offers an empirical valuation of mortgages that is more easily implemented than the structural-form model. Because of these reasons, we chose the reduced-form model to value a mortgage.

We evaluate a fully amortized fixed-rate mortgage (FRM), which has initial principal balance $M(0)$, fixed coupon rate c and a time to maturity of T years based on a continuous-time framework. The definitions of mortgage cash flow are expressed in the following. If a default or prepayment event does not occur, the continuous payout Y and the mortgage balance $M(t)$ at time t can be obtained as follows:

$$Y = M(0) \times \frac{c}{1-e^{-cT}}, \text{ and } M(t) = M(0) \times \frac{1-e^{-c(T-t)}}{1-e^{-cT}}. \quad (1)$$

For the descriptions of the mortgage borrower's termination behavior, we adopt the model shown in Tsai and Chiang (2012). With this valuation model, if the prepayment occurs at a random time τ_P within the range t to T, the cash payment obtained by the lender is $M(\tau_P)$. If the default occurs at a random time τ_D within the range t to T, the cash payment received by the lender is $M(\tau_D)\rho(\tau_D)$, where $\rho(\tau_D)$ is the recovery rate at τ_D, with $0 < \rho(\tau_D) < 1$. We let $\theta(t)$ and $\pi(t)$ be the hazard rates of a loan being prepaid and defaulted respectively at time t. Under no arbitrage and a complete market, the mortgage value at time t, denoted as $V(t)$, can be expressed as follows (see Bielecki and Rutkowski, 2002; Liao et al., 2008; Tsai and Chiang, 2012):

$$V(t) = Y\int_t^T E[e^{-\int_t^s (r(u)+\theta(u)+\pi(u))du}]ds + \int_t^T M(s)E[\theta(s)e^{-\int_t^s (r(u)+\theta(u)+\pi(u))du}]ds$$

$$+\int_t^T M(s)E[\rho(s)\pi(s)e^{-\int_t^s (r(u)+\theta(u)+\pi(u))du}]ds\,, \tag{2}$$

where $E[\cdot]$ is an expected operator at time t under a risk-neutral measure. The first part of the right side of Equation (2) specifies the expected value of a mortgage that does not terminate until maturity (hereafter denoted as the survival value). The second and the third parts respectively represent the expected values of a mortgage that has been prepaid and defaulted before maturity (hereafter denoted as prepayment value and default value, respectively).

2.2 Deriving a closed-form formula for a mortgage value using a Lévy model

Results from previous researches indicate that the termination probabilities and the recovery rate are significantly dependent on the state variables such as the interest rate and the housing return rate (Green and Shoven, 1986; Schwartz and Torous, 1989, 1993; Smith et al., 1996; Lambrecht et al., 2003; Dermine and de Carvalho, 2006). Because several studies (e.g., Sutton, 2002; Borio and Mcguire, 2004, and Tsatsaronis and Zhu, 2004) indicate that there is a significant correlation between the interest rate and the housing return rate, their correlation is also incorporated in our mortgage valuation model. We therefore specify the prepayment and default hazard rates as well as the recovery rate to be affine functions of the correlated state variables, including the interest rate and the housing return rate. They are:

$$\theta(t) = \varphi_0^\theta(t) + \varphi_r^\theta r(t) + \varphi_H^\theta r_H(t)\,, \tag{3}$$

$$\pi(t) = \varphi_0^\pi(t) + \varphi_r^\pi r(t) + \varphi_H^\pi r_H(t)\,, \text{ and} \tag{4}$$

$$\rho(t)=\varphi_0^\rho(t)+\varphi_r^\rho r(t)+\varphi_H^\rho r_H(t), \tag{5}$$

where

$r(t)$ is the default-free short-term interest rate;

$r_H(t)$ is the housing return rate;

$\varphi_0^l(t)$ is the baseline hazard rate of prepayment, the baseline hazard rate of default and the baseline recovery rate (given $l=\theta,\pi,\rho$ respectively) at time t; and

φ_r^l and φ_H^l denote the coefficients of the interest rate and the housing return rate, respectively.

We then show how to specify the interest rate and the housing return rate as time-changed Lévy processes. Here we let $r(t)$ and $r_H(t)$ be specified to follow Lévy processes of the Ornstein-Uhlenbeck (OU) type. In such specifications, the interest rate and the housing return rate are composed of two parts: one part is the drift term specified in the OU process, and the other part is the Lévy portion. They can be represented by the following differential equations (Barndorff-Nielsen and Shephard, 2001):

$$dr(t)=a_r(a_r^{-1}\theta_r(t)-r(t))dt+dL_r(t), \text{ and} \tag{6}$$

$$dr_H(t)=a_H(a_H^{-1}\theta_H(t)-r_H(t))dt+\rho_{r,H}dL_r(t)+dL_H(t), \tag{7}$$

where a_r and a_H are the adjusted speeds of the interest rate and the housing return rate, respectively; $a_r^{-1}\theta_r(t)$ and $a_H^{-1}\theta_H(t)$ are the long-term interest rate and the long-term housing return rate. $dL_r(t)$ and $dL_H(t)$ are assumed to be Lévy processes under a risk-neutral measure. We assume that they satisfy the three conditions of a Lévy process.[2] In

[2] According to the definitions in Schoutens (2003), an adapted process $X(t)=(X(s))_{t\le s<\infty}$ is a Lévy process if the following three conditions are satisfied: (1) $X(t)$ has increments independent of the past; (2) $X(t)$ has

addition, $dL_r(t)$ and $dL_H(t)$ are assumed to be independent. The correlation between the processes of the interest rate and the housing return rate is denoted as the variable $\rho_{r,H}$, where $\rho_{r,H} = \dfrac{Cov(dr(t), dr_H(t))}{Var(dL_r(t))}$, Cov and Var are the symbols for covariance and variance, respectively.

Since we use a Lévy model with an infinite-activity jump process, $dL_i(t)$ ($i = r$ and H) is assumed to be a type of subordinator.[3] Thus, according to Equation (6), we have the following:

$$r(s) = r^{\mu}(s) + \int_t^s e^{-a_r(s-v)} dL_r(v), \text{ and} \tag{8}$$

$$\Xi_r(t,T) = \Xi_r^{\mu}(t,T) + \int_t^T \eta_r(v) dL_r(v), \tag{9}$$

where

$r^{\mu}(s) = r(t)e^{-a_r(s-t)} + \int_t^s e^{-a_r(s-v)} \theta_r(v) dv$;

$\Xi_r(t,T)$ is the cumulated interest rate and $\Xi_r(t,T) = \int_t^T r(s) ds$;

$\Xi_r^{\mu}(t,u) = r(t)\eta_r(u) + \psi(t,u)$;

$\eta_r(v) = a_r^{-1}(1 - e^{-a_r(v-t)})$; and

$\psi(t,u) = \int_t^u (\int_t^s e^{-a_r(v-t)} \theta(v) dv) ds$.

Moreover, the cumulant of the cumulated interest rates $\Xi_r(t,T)$ is expressed as follows:

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stationary increments; and (3) $X(t)$ is continuous in probability.

[3] According to the Lévy-Khintchine formula, the characteristic function (denoted as $\phi_X(b)$) for each Lévy process with triple $(\gamma, \zeta^2, \nu(dx))$, where $\gamma \in R$, $\zeta > 0$ and $\nu(\cdot)$, is a measure of $R \setminus \{0\}$, such that $\int_R (1 \wedge x^2) \nu(dx) < \infty$ can be expressed as follows:

$$\phi_X(b) = \exp(i\gamma b - \frac{\zeta^2}{2} b^2 + \int_R (1 - e^{ibx} + ibxI_{\{|x|<1\}}) \nu(dx)),$$

where $i = \sqrt{-1}$, $b \in R$ and $I_{\{\}}$ is an indicated function, $I_{\{\omega\}} = 1$, if state ω occurs, otherwise $I_{\{\omega\}} = 0$. The subordinator is defined as $\gamma = 0$, $\zeta^2 = 0$ and $\nu(dx)$ is defined on $(0, \infty)$ in the triple $(\gamma, \zeta^2, \nu(dx))$ of the Lévy process.

$$\kappa_r^{\Xi}(b,t,u) \equiv \ln(E[\exp(b\Xi_r(t,u))]) = b\Xi_r^{\mu}(t,u) + \int_t^u \kappa_r(b\eta_r(s))ds, \tag{10}$$

where $\kappa_r(b) = \log(E[\exp(bdL_r)])$ is the cumulant of dL_r.

Likewise, according to Equation (7), we have the following:

$$r_H(s) = r_H^{\mu}(s) + \int_t^s e^{-a_H(s-v)}\rho_{r,H}dL_r(v) + \int_t^s e^{-a_H(s-v)}dL_H(v), \text{ and} \tag{11}$$

$$\Xi_H(t,u) = \Xi_H^{\mu}(t,u) + \int_t^u \eta_H(v)\rho_{r,H}dL_r(v) + \int_t^u \eta_H(v)dL_H(v), \tag{12}$$

where

$r_H^{\mu}(s) = r_H(t)e^{-a_H(s-t)} + \int_t^s e^{-a_H(s-v)}\theta_H(v)dv$;

$\Xi_H(t,u)$ is the cumulated housing return rate and $\Xi_H(t,u) = \int_t^u r_H(s)ds$;

$\Xi_H^{\mu}(t,u) = r_H(t)\eta_H(u) + \psi_H(t,T)$;

$\eta_H(v) = a_H^{-1}(1 - e^{-a_H(v-t)})$; and

$\psi_H(t,T) = \int_t^T (\int_t^u e^{-a_H(v-t)}\theta_H(v)dv)du$.

In addition, the cumulant of cumulated housing return rate $\Xi_H(t,T)$ can be expressed as follows:

$$\begin{aligned}\kappa_H^{\Xi}(b,t,T) &= \ln(E[\exp(b\Xi_H(t,T))]) \\ &= b\Xi_H^{\mu}(t,T) + \int_t^T \kappa_H(b\eta_H(s))ds + \int_t^T \kappa_r(b\rho_{r,H}\eta_H(s))ds,\end{aligned} \tag{13}$$

where $\kappa_H(b) = \log(E[\exp(bdL_H)])$ is the cumulant of dL_H.

According to Equations (3), (4), (5), (9) and (12), we have:

$$\int_t^u (r(s) + \theta(s) + \pi(s))ds \equiv A_I(t,u) + A'\Xi(t,u), \tag{14}$$

where $A_I(t,u) = \int_t^u (\varphi_0^{\theta}(s) + \varphi_0^{\pi}(s))ds$; $A = [A_r \quad A_H]$, $A_r = 1 + \varphi_r^{\theta} + \varphi_r^{\pi}$; $A_H = \varphi_H^{\theta} + \varphi_H^{\pi}$; and $\Xi(t,u) = [\Xi_r(t,u) \quad \Xi_H(t,u)]$.

Based on our preceding specifications and the above derived results, the valuation of the mortgage in Equation (2) can be easily determined from the moment-generating functions (MGF), the cumulants for the processes of interest rate and the housing return rate. The MGF of $-A_I(t,T)-A'\Xi(t,T)$ (denoted as $\Psi(t,u)$) is the expected survival probability of the mortgage. It can be expressed as follows (see Appendix A):

$$\begin{aligned}\Psi(t,u) &\equiv E[\exp(-\int_t^u (r(s)+\theta(s)+\pi(s))ds)] \\ &= E[\exp(-A_I(t,T)-A'\Xi(t,u))] \\ &= \exp(-A_I(t,T)+\overline{\Xi}_r(-A_r,t,u)+\kappa_r^{\Xi}(-A_r,-A_H,t,u) \\ &\quad +\overline{\Xi}_H(-A_H,t,u)+\kappa_H^{\Xi}(-A_H,t,u))),\end{aligned} \tag{15}$$

where

$\overline{\Xi}_r(-A_r,t,u)=-A_r\Xi_r^{\mu}(t,u)$;

$\kappa_r^{\Xi}(-A_r,-A_H,t,u)=\int_t^u \kappa_r(-A_r\eta_r(s)-A_H\rho_{r,H}\eta_H(s))ds$;

$\overline{\Xi}_H(-A_H,t,u)=-A_H\Xi_H^{\mu}(t,u)$; and

$\kappa_H^{\Xi}(-A_H,t,u)=\int_t^u \kappa_H(-A_H\eta_H(s))ds$.

The formula of $E_t[\theta(u)\exp(-\int_t^u (r(s)+\theta(s)+\pi(s))ds)]$ (denoted as $\Psi^P(t,u)$) is the prepayment probability of the mortgage. It can be shown in the following (see Appendix A):

$$\Psi^P(t,u)=\theta^*(u)\times\Psi(t,u), \tag{16}$$

where

$$\begin{aligned}\theta^*(u) &= \varphi_0^{\theta}+\varphi_r^{\theta}r^{\mu}(u)+\varphi_H^{\theta}r_H^{\mu}(u) \\ &\quad +(\varphi_r^{\theta}\eta_r(u)+\varphi_H^{\theta}\eta_H(u)\rho_{r,H})\kappa_r'(-A_r\eta_r(u)-A_H\rho_{r,H}\eta_H(u)) \\ &\quad +\varphi_H^{\theta}\eta_H(u)\kappa_H'(A_H\eta_H(u));\ \text{and}\end{aligned}$$

$$\kappa_i'(b) = \frac{\partial \kappa_i(b)}{\partial b}, \text{ for } i = r, H.$$

The formula of $E_t[\pi(u)\rho(u)\exp(-\int_t^u (r(s)+\theta(s)+\pi(s))ds)]$ (denoted as $\Psi^D(t,u)$) is the expected value of the recovery rate given default. We have (see Appendix A):

$$\Psi^D(t,u) = (\pi^*(u)\rho^*(u) + \Psi^{\pi,\rho}(t,u)) \times \Psi(t,u), \tag{17}$$

where

$$\pi^*(u) = \varphi_0^\pi + \varphi_r^\pi r^\mu(u) + \varphi_H^\pi r_H^\mu(u)$$

$$+ (\varphi_r^\pi \eta_r(u) + \varphi_H^\pi \eta_H(u)\rho_{r,H})\kappa_r'(-A_r\eta_r(u) - A_H\eta_H(u)\rho_{r,H})$$

$$+ \varphi_H^\pi \eta_H(u)\kappa_H'(-A_H\eta_H(u));$$

$$\rho^*(u) = \varphi_0^\rho + \varphi_r^\rho r^\mu(u) + \varphi_H^\rho r_H^\mu(u)$$

$$+ (\varphi_r^\rho \eta_r(u) + \varphi_H^\rho \eta_H(u)\rho_{r,H})\kappa_r'(-A_r\eta_r(u) - A_H\eta_H(u)\rho_{r,H})$$

$$+ \varphi_H^\rho \eta_H(u)\kappa_H'(-A_H\eta_H(u));$$

$$\Psi^{\pi,\rho}(t,u) = [\varphi_r^\pi \eta_r(u) + \varphi_H^\pi \eta_H(u)\rho_{r,H})\ (\varphi_r^\rho \eta_r(u) + \varphi_H^\rho \eta_H(u)\rho_{r,H})$$

$$\times \kappa_r''(-A_r\eta_r(u) - A_H\eta_H(u)\rho_{r,H})]$$

$$+ (\varphi_H^\pi \eta_H(u))(\varphi_H^\rho \eta_H(u)) \times \kappa_H''(-A_H\eta_H(u)); \text{ and}$$

$$\kappa_i''(b) = \frac{\partial^2 \kappa_i(b)}{\partial b^2} \text{ for } i = r, H.$$

Substituting Equations (15)-(17) into Equation (2), we obtain the pricing formula for the mortgage value. It is:

$$V(t) = \int_t^T (Y\Psi(t,u) + M(u)\Psi^P(t,u) + M(u)\Psi^D(t,u))du. \tag{18}$$

The formula in Equation (18) is a general mortgage valuation model. It includes the prepayment and default hazard rates and the recovery rate as multivariate affine functions that include the correlated stochastic state variables (i.e., interest rate and the housing return rate) following a time-changed Lévy process. Based on our pricing formula, one can use different types of Lévy model to value a mortgage. The following section introduces three types of Lévy model for this application.

3. Our Model's Application for the Different Types of Lévy Model

The pricing formula for each Lévy models can be obtained if its cumulant ($\kappa_i(b)$, for $i = r, H$) and derivatives of the cumulantare obtained. In the following subsections, we express the MGFs, the cumulants and their derivatives of the cumulant for the normal, VG and NIG models. For simplify, we ignore the subscript of $\kappa_i(b)$ and denote $\kappa_i(b)$ in normal model, VG model and NIG model as $\kappa(b \mid 0, \sigma_N^2)$, $\kappa(b \mid \theta_{VG}, \nu_{VG}, \sigma_{VG})$ and $\kappa(b \mid \theta_{NIG}, \nu_{NIG}, \sigma_{NIG})$, respectively.

3.1 Normal model

To derive the mortgage valuation, traditional studies have specified the processes of interest rates and housing return rate by BM (Kau, Keenan and Smurov, 2004; Liao, Tsai, Chiang, 2008). Here we first illustrate the specification of the BM process. BM with drift is a Lévy process that has gaussian increments. We denote $dX^N(t)$ as a normal process. The probability density function (PDF) for $dX^N(t)$ is defined as follows:

$$f(x \mid 0, \sigma_N^2) = \frac{1}{\sqrt{2\pi\sigma_N^2}} e^{-\frac{x^2}{2\sigma_N^2}}, \tag{19}$$

where σ_N is the standard deviation of the normal distribution. The MGF (denoted as $\phi_N(b \mid 0, \sigma_N^2)$) and the cumulant for the normal process are given respectively by

$$\phi_N(b \mid 0, \sigma_N^2) = \exp(\frac{b^2\sigma_N^2}{2}), \text{ and} \tag{20}$$

$$\kappa(b\,|\,0,\sigma_N^2) = \frac{b^2\sigma_N^2}{2}. \tag{21}$$

The first-order derivative and the second-order derivative of the cumulant can be expressed respectively as follows:

$$\kappa'(b\,|\,0,\sigma_N^2) = \frac{\partial \kappa(b)}{\partial b} = b\sigma_N^2, \text{ and} \tag{22}$$

$$\kappa''(b\,|\,0,\sigma_N^2) = \frac{\partial^2 \kappa(b)}{\partial b^2} = \sigma_N^2. \tag{23}$$

From these results we get $\kappa_r(b\,|\,0,\sigma_N^2)$, $\kappa_r'(b\,|\,0,\sigma_N^2)$ and $\kappa_r''(b\,|\,0,\sigma_N^2)$ for the interest rate and $\kappa_H(b\,|\,0,\sigma_N^2)$, $\kappa_H'(b\,|\,0,\sigma_N^2)$ and $\kappa_H''(b\,|\,0,\sigma_N^2)$ for the housing return rate. Then, applying these results in Equation (18) we obtain the explicit formula for mortgage valuation under the assumption of a normal model of the state variables.

3.2 VG model

The VG model was developed by Madan and Seneta (1990) and extended to incorporate skewness by Madan and Milne (1991) and Madan et al. (1998). Several studies document that the VG model performs better than other Lévy models for financial data series (Madan et al., 1998; Daal and Madan, 2005). The VG model is interpreted as BM with drift, where time is changed by a gamma process. It is one of the most popular infinite-activity models with finite variation but relatively low activity of small jumps. This process is a pure jump process with an infinite arrival rate of jumps, but unlike BM (that also has infinite motion) the process has finite variation and can be written as the difference of two increasing processes, each giving separately the market up and down moves.

The PDF for a gamma distribution is defined as follows:

$$f_G(x\,|\,\alpha_G,\beta_G) = \frac{\beta_G^{\alpha_G}}{\Gamma(\alpha_G)} x^{\alpha_G-1} e^{-x\beta_G}, \tag{24}$$

where $\Gamma(\cdot)$ is a gamma function, and α_G and β_G are the parameters of the gamma distribution.

The VG process is obtained by evaluating arithmetic BM with drift θ_{VG} and volatility σ_{VG} at random times given by a gamma process having a mean rate per unit time of 1 and a variance rate of v_{VG} (Carr and Madan, 1998). The VG process (denoted as $X^{VG}(t)$) with parameters $\theta_{VG}, v_{VG}, \sigma_{VG}$ is defined as follows:

$$X^{VG}(t) = \theta_{VG} G(t) + \sigma_{VG} W_{G(t)}, \tag{25}$$

where $G(t)$ is a gamma process with parameters $\alpha_G = \frac{1}{v_{VG}} > 0$ and $\beta_G = \frac{1}{v_{VG}}$. The VG process has three parameters: (1) σ_{VG} *is* the volatility of the BM, (2) v_{VG} is the variance of the gamma time change and (3) θ_{VG} is the drift in the process.

The MGF[4] (denoted as $\phi_{VG}(b \mid \theta_{VG}, v_{VG}, \sigma_{VG})$) and the cumulant for the VG model are given respectively by

$$\phi_{VG}(b \mid \theta_{VG}, v_{VG}, \sigma_{VG}) = (1 + b\theta_{VG} v_{VG} + \frac{1}{2}\sigma_{VG}^2 v_{VG} b^2)^{-\frac{1}{v_{VG}}}; \text{ and} \tag{26}$$

$$\kappa(b \mid \theta_{VG}, v_{VG}, \sigma_{VG}) = -\frac{1}{v_{VG}} \ln(1 + b\theta_{VG} v_{VG} + \frac{1}{2}\sigma_{VG}^2 v_{VG} b^2). \tag{27}$$

The first-order and second-order derivatives for the cumulant can be expressed respectively as follows:

$$\begin{aligned}&\kappa'(b \mid \theta_{VG}, v_{VG}, \sigma_{VG}) \\ &\quad = -\frac{1}{v_{VG}}(1 + b\theta_{VG} v_{VG} + \frac{1}{2}\sigma_{VG}^2 v_{VG} b^2)^{-1} \times (\theta_{VG} v_{VG} + \sigma_{VG}^2 v_{VG} b_{VG}); \text{ and}\end{aligned} \tag{28}$$

$$\begin{aligned}&\kappa''(b \mid \theta_{VG}, v_{VG}, \sigma_{VG}) \\ &\quad = \frac{1}{v_{VG}}(1 + b\theta_{VG} v_{VG} + \frac{1}{2}\sigma_{VG}^2 v_{VG} b^2)^{-2} \times (\theta_{VG} v_{VG} + \sigma_{VG}^2 v_{VG} b)^2 \\ &\qquad - \frac{1}{v_{VG}}(1 + b\theta_{VG} v_{VG} + \frac{1}{2}\sigma_{VG}^2 v_{VG} b^2)^{-1} \times (\sigma_{VG}^2 v_{VG}).\end{aligned} \tag{29}$$

[4] The MGFs for the VG and NIG processes are obtained from their characteristic functions.

Using the formulas shown in Equations (27)–(29), we get $\kappa_i(b\,|\,\theta_{VG},\nu_{VG},\sigma_{VG})$, $\kappa_i'(b\,|\,\theta_{VG},\nu_{VG},\sigma_{VG})$, and $\kappa_i''(b\,|\,\theta_{VG},\nu_{VG},\sigma_{VG})$ for $i=r,H$; and then one can use these to obtain the explicit valuation formula for the mortgage based on Equation (18) under the assumption that the state variables follow a VG process.

3.3 NIG model

The NIG model was developed and has been discussed by Eberlein and Keller (1995), Barnriorff-Nielsen (1995, 1997a,b, 1998) and Rydberg (1997b), amongst others. Barndorff-Nielsen (1995), Rydberg (1997b), Seneta (2004), Daal and Madan (2005) and Figueroa-Lopez et al. (2011) demonstrate that the NIG model is superior to the normal model for the goodness-of-fit of return distributions. It is one of the most popular infinite-activity models with infinite variation in any finite interval of time, and it belongs to the class of generalized hyperbolic Lévy processes (Schoutens, 2003; Cont and Tankov, 2004). This process captures the asymmetry and leptokurtic nature of the underlying rate distribution. It is defined as an inverse Gaussian (IG) time-changed BM with drift. The IG process is usually used to describe a random first time at which BM reaches a positive level. The PDF for the IG is defined as follows:

$$f_{IG}(x\,|\,\alpha_{IG},\beta_{IG})=\frac{\alpha_{IG}}{\sqrt{2\pi}}x^{-\frac{2}{3}}e^{\alpha_{IG}\beta_{IG}-\frac{1}{2}(\frac{\alpha_{IG}^2}{x}+\beta_{IG}^2x)}, \tag{30}$$

where α_{IG} and β_{IG} are the parameters of the IG distribution.

We denote the NIG distribution as $NIG(\theta_{NIG},\nu_{NIG},\sigma_{NIG})$. The NIG process can be expressed as

$$X^{NIG}(t)=\theta_{NIG}\sigma_{NIG}^2I(t)+\sigma_{NIG}W_{I(t)}, \tag{31}$$

where $I(t)$ is an IG process with parameters $\alpha_{IG}=1$ and $\beta_{IG}=\sigma\sqrt{\nu_{NIG}^2-\theta_{NIG}^2}$. The MGF (denoted as $\phi_{NIG}(b\,|\,\theta_{NIG},\nu_{NIG},\sigma_{NIG})$) and the cumulant for the NIG model are given respectively (Barndorff-Nielsen, 1998):

$$\phi_{NIG}(b\,|\,\theta_{NIG},\nu_{NIG},\sigma_{NIG}) = \exp(-\sigma_{NIG}(\sqrt{\nu_{NIG}^2-(\theta_{NIG}+b)^2}-\sqrt{\nu_{NIG}^2-\theta_{NIG}^{\ 2}}))\text{; and} \qquad (32)$$

$$\kappa(b\,|\,\theta_{NIG},\nu_{NIG},\sigma_{NIG}) = -\sigma_{NIG}(\sqrt{\nu_{NIG}^2-(\theta_{NIG}+b)^2}-\sqrt{\nu_{NIG}^2-\theta_{NIG}^{\ 2}})\,. \qquad (33)$$

The first-order and the second-order derivatives of the cumulant can be expressed as follows:

$$\kappa'(b\,|\,\theta_{NIG},\nu_{NIG},\sigma_{NIG}) = \sigma_{NIG}(\theta_{NIG}+b)(\nu_{NIG}^2-(\theta_{NIG}+b)^2)^{-\frac{1}{2}}\text{; and} \qquad (34)$$

$$\kappa''(b\,|\,\theta_{NIG},\nu_{NIG},\sigma_{NIG})$$

$$= \sigma_{NIG}(\nu_{NIG}^2-(\theta_{NIG}+b)^2)^{-\frac{1}{2}} + \frac{\sigma_{NIG}}{2}(\theta_{NIG}+b)^2(\nu_{NIG}^2-(\theta_{NIG}+b)^2)^{-\frac{3}{2}}\,. \qquad (35)$$

Applying the above results one can obtain $\kappa_i(b\,|\,\theta_{NIG},\nu_{NIG},\sigma_{NIG})$, $\kappa_i'(b\,|\,\theta_{NIG},\nu_{NIG},\sigma_{NIG})$ and $\kappa_i''(b\,|\,\theta_{NIG},\nu_{NIG},\sigma_{NIG})$, for $i = r, H$; and then through Equation (18) the explicit valuation formula for a mortgage under the assumption of the NIG model for the state variables can be obtained.

4. Estimation Method

Here a two-step method is adopted for the estimation. We first use the OLS method to estimate the drift terms for the two processes of the interest rate and the housing return rate. The second step is to use the empirical likelihood method to estimate the essential parameters when the two state variables are specified as one of the three Lévy models.

To begin, we explain how to estimate the drift terms of the two processes. We let $dt = 1$. According to Equations (6) and (7), we have the following respectively:

$$dr(t) = \theta_r(t) - a_r r(t-1) + dL_r(t)\text{; and} \qquad (36)$$

$$dr_H(t) = \theta_H(t) - a_H r(_H t-1) + \rho_{r,H} dL_r(t) + dL_H(t)\,. \qquad (37)$$

For simplification, we let $\theta_r(t)$ and $\theta_H(t)$ be the functions of the time, which are respectively $\theta_r(t)=\theta_r^0+\theta_r^1 t$ and $\theta_H(t)=\theta_H^0+\theta_H^1 t$. We then obtain the following regressions:

$$dr(t)=\theta_r^0+\theta_r^1 t-a_r r(t-1)+\varepsilon_r(t)\text{, and} \tag{38}$$

$$dr_H(t)=\theta_H^0+\theta_H^1 t-a_H r_H(t-1)+\varepsilon_H(t)\,. \tag{39}$$

The OLS method is used to estimate the parameter values (i.e., $\hat{\theta}_r^0$, $\hat{\theta}_r^1$, $\hat{a}_r$, $\hat{\theta}_H^0$, $\hat{\theta}_H^1$ and $\hat{a}_H$) of the above two regressions.

Next, we illustrate how to estimate the parameters of the Lévy process. To estimate the parameters for the Lévy process of the interest rate, we let $d\hat{L}_r(t)=\varepsilon_r(t)$, for which $d\hat{L}_r(t)$ represents the sample data for $dL_r(t)$. To estimate the parameters for the Lévy process of the housing return rate, we let $d\hat{L}_H(t)=\varepsilon_H(t)-\hat{\rho}_{r,H}\varepsilon_r(t)$, where $d\hat{L}_H(t)$ represents the sample data for $dL_H(t)$, and $\hat{\rho}_{r,H}=\dfrac{Cov(dr(t),dr_H(t))}{Var(\varepsilon_r(t))}$. The data of $d\hat{L}_r(t)$ and $d\hat{L}_H(t)$ are used to estimate the parameters for the Lévy models, shown in Section 3, adopting the maximum empirical likelihood method (Qin and Lawless, 1994; Elgin, 2011), an introduction to which is provided in Appendix B.

5. Implementations of the Valuation Model

In this section, we use real mortgage data to show how one can apply our model to obtain the mortgage value and its duration. In the first subsection, we provide a data description for each variable used in our model. The second subsection provides discussion of how the assumptions of the termination rate influence the mortgage value and its duration, which is due to the changes in the interest rate in these three Lévy models.

5.1 Data description

We use the data obtained from Tsai and Chiang (2012) to implement the model. In this data, the monthly prepayment probability, default probability and recovery rate were taken from the CoreLogic LoanPerformance Securities Database.[5] The data includes first-lien mortgages issued from 2001 to 2008 at a 30-year fixed rate. The housing prices were the Standard and Poor's Case-Shiller 10-City Home Price Index. For the short-term interest rates, we adopted the interest rates for a 3-month U.S. treasury bill. The sample period from September 2001 to October 2010 yielded 110 observations for each variable. Table 1 presents descriptive information on the sample data: the mean, standard deviation, median, and maximum and minimum values of each short-term interest rate, housing price index, prepayment probability, default probability and recovery rate. The results show that the mean prepayment probability is 8.7 times the mean of the default probability (0.2146/0.0248), and thus one can show that the mortgage termination was caused mainly by the prepayment risk.

< Insert Table 1 Here>

5.2 Empirical results

Because we used the data from Tsai and Chiang (2012), we refer to their results for the coefficients of the affine functions, the prepayment and default hazard rates and the recovery rate. The coefficients of the three affine functions (i.e., Equations (3)–(5)) are φ_0^θ =0.3812, φ_r^θ = -4.6498, φ_H^θ = 0.7846, φ_0^π = 0.0424, φ_r^π = -0.8846, φ_H^π = -0.0846, φ_0^ρ = 0.4017, φ_r^ρ = 3.7656 and φ_H^ρ = 1.3112.

We use the OLS method to estimate the parameters of the OU process of the short-term interest rate and housing price processes. The estimates are shown in Table 2. According to

[5] The recovery rate was derived by annualizing the monthly mortality data from the CoreLogic Company.

the empirical results from Equations (38) and (39), the estimated parameters for the interest rate are $\theta_r^0 = 2.9259\times10^{-4}$, $\theta_r^1 = -1.8492\times10^{-6}$ and $a_r = -8.8384\times10^{-3}$; none is significant at the 10% level. The estimates for the housing return rate are $\theta_H^0 = 1.5243\times10^{-3}$, $\theta_H^1 = -5.1887\times10^{-6}$ and $a_H = 0.34354$; all are significant at the 10% level. For both processes, all the time trends are negative. Thus, the time trends are downward for the long-term interest rate and the long-term housing return rate. The correlation between the interest rate and the housing return rate is calculated using Equations (6) and (7), from which we obtain $\rho_{r,H} = 0.0343$.

< Insert Table 2 Here >

The residuals of the regressions (i.e., Equations (38) and (39)) are used to estimate the parameters of the three Lévy models for the interest rate and the housing return rate respectively. We use the empirical likelihood method to estimate the parameters in the Lévy models. To perform the estimation, we need the initial values for each parameter in the Lévy models. We use the moment method to obtain these initial values. In Appendix C, we show the moment functions for the three Lévy models (i.e., normal, VG and NIG). We also show the estimates for the parameters of the short-term interest rate and housing price processes for each Lévy model using these moment functions.

After obtaining the initial values of the parameters, we use the maximum empirical likelihood method to estimate the parameters of the short-term interest rate and housing price processes in each Lévy model. The results are shown in Table 3. For the normal model, the estimates are $\sigma_N = 0.0053$ for the interest rate and $\sigma_N = 0.0071$ for the housing return rate. For the VG model, the estimates are $\sigma_{VG} = 0.0054$, $\nu_{VG} = 257.3063$ and $\theta_{VG} = 1.8292\times10^{-6}$

for the interest rate; and σ_{VG}= 0.0074, ν_{VG}= 278.5796 and θ_{VG}= 3.1496×10^{-7} for the housing return rate. For the NIG model, the estimates are σ_{NIG} = 6.3216×10^{-10}, ν_{NIG} =37.2515 and θ_{NIG} =-0.8033 for the interest rate; and σ_{NIG}= 9.5083×10^{-5}, ν_{NIG} = 0.9469 and θ_{NIG} = -0.0532 for the housing return rate. Most of these estimates are significant at the 1% level.

To determine which model best fits the actual underlying distributions, we refer to the maximum empirical log-likelihood ratios ($l_n(\xi)$). For the interest rate, the values of $l_n(\xi)$ are 5.8391×10^{-4}, 8.1119×10^{-4} and 7.6876×10^{-6} for the normal, VG and NIG models respectively. For the housing return rate, the values of $l_n(\xi)$ are 7.8589×10^{-4}, 7.8628×10^{-4} and 5.8972×10^{-4} for the normal, VG and NIG models respectively. The VG model has the largest maximum empirical log-likelihood ratios no matter whether the estimates are for the interest rate or the housing return rate. Thus, the VG model is better than the other two models for capturing the residual paths of the interest rate and the housing return rate. Thus, mortgage valuation using the VG model should be better than that using the normal and NIG models. Accordingly, one can infer that the drifts of the residuals for the interest rate and the housing return rate are well interpreted as BM, where the time is changed by a gamma process.

< Insert Table 3 Here>

To calculate the mortgage values, we enter the parameter values into the mortgage valuation model as follows: the basic parameter values ($M(0)=\$100$, $c=10\%$, $r(0)=3\%$, and $r_H(0)=2\%$), the estimates of the parameters for the drift terms of the OU processes for the interest rate and the housing return rate (see Table 2) and the estimates of the Lévy model parameters of the interest rate and the housing return rate (see Table 3). The calculated mortgage values are shown in Table 4. The mortgage values are \$101.2349, \$103.3564 and \$103.4505 for the Lévy process respectively specified as the normal, VG and NIG models. Our numerical results show that this valuation framework can be utilized in practical applications.

< Insert Table 4 Here>

The numerical results in Table 4 show that the mortgage value estimated by the normal model has the lowest value among the three Lévy models. Moreover, based on our model, one can extensively analyze the mortgage value to reveal its components, such as the survival value, the prepayment value and the default value. With the normal model, the survival value is \$31.8618, the prepayment value is \$67.7755 and the default value is \$1.5976. With the VG model, the survival value is \$30.4477, the prepayment value is \$70.6049 and the default value is \$2.3038. With the NIG model, the survival value is \$30.7979, the prepayment value is \$70.3599 and the default value is \$2.2927. Accordingly, we find the prepayment value is the largest among three parts of the mortgage value, no matter which Lévy model is used. We can infer that the prepayment risk plays a key role in the valuation of a mortgage contract. Moreover, the termination (i.e., the prepayment and default) value in the normal model is lower than in the VG and NIG models. The main reason may come from that the normal model does not do a good job in capturing the real shapes of the termination hazard rates and the recovery rate given default.

5.3 Discussion of the mortgage value and duration based on our model

In traditional mortgage valuation, the termination rate (i.e., prepayment or default) is usually assumed to be a deterministic or constant value. For example, the PSA prepayment model,[6]

[6] The 100% PSA works as follows: the prepayment rate is 0.2% in the first month and then increases 0.2% each month until it reaches 6% in the thirtieth month. From the thirtieth month on, the prepayment rate is assumed to be 6%.

developed by the Public Securities Association, is commonly used for pricing mortgage-backed securities. This model let the prepayment rate be deterministic values. However, assuming a deterministic termination rate, which is not influenced by the related variables (e.g., interest rate), may cause unreasonable pricing results for mortgage-related securities. In Figures 2a and 2b, we respectively show the mortgage value and its duration corresponding to the different initial value of the interest rate when the termination hazard rate and recovery rate are not influenced by the state variables. To illustrate this, we let the coefficients for the affine functions of the interest rate and the housing return rate be zero. Specifically, $\varphi_i^{\theta}= 0$, $\varphi_i^{\pi}= 0$ and $\varphi_i^{\rho}= 0$, for $i = r, H$. Moreover, we let $r(0)$ range from 0.02 to 0.07. In Figure 2a, similar to analyses of a bond contract, there is a negative relationship between the mortgage value and interest rate. Moreover, Figure 2b shows that the duration of the mortgage is a positive value and is negatively correlated with the interest rate.

< Insert Figure 2 Here>

In Figures 3a and 3b, we use our model to respectively show the mortgage value and its duration corresponding to the different interest rates. In these figures, the solid line represents the estimated values under the normal model, the dotted represents the estimated values under the VG model and the dashed line represents the estimated values under the NIG model. Figure 3a tells us two interesting things. First, there is a positive relationship between the mortgage value and interest rate. This result may come from the fact that we let $a_r^{\theta}=$ -4.6498 on our numerical example. Thus, a decrease in the interest rate greatly increases the prepayment rate and then substantially decreases the survival value of the mortgage. As just mentioned, the prepayment risk plays a key role in the valuation of a mortgage contract. Therefore, the relationship between the mortgage value and the initial interest rate may no longer be negative, as shown in Figure 2. Our results reveal that specifying the termination hazard rates, which are not influenced by the interest rate, could result in an unreasonable mortgage value. Since previous studies demonstrate that the prepayment rate is greatly influenced by the interest rate (Schwartz and Torous, 1993; Tsai and Chiang, 2012), one should consider the fact that the termination hazard rate is influenced by the state variables in valuing a mortgage contract.

< Insert Figures 3a and 3b Here>

Second, Figure 3a shows that the mortgage value estimated by the Lévy model with the normal process has the lowest estimated value among the three Lévy models. The same results can also be found in Table 4. As we justly mentioned, the main reason likely is that the normal model ignores the excess kurtosis, skewness, fat tails and jump effects in the real processes of the interest rate and the housing return rate. Thus, if one ignores the above facts on the actual processes of the state variables, the mortgage value could be underestimated. Moreover, our preceding results show that the VG model captures the processes of the interest rate and the housing return rate better than the NIG model. Thus, the pricing results should be better with the VG model than with either the normal or NIG models.

Figure 3b shows the mortgage durations corresponding to the different interest rates. The results show that the duration is also negatively correlated with the interest rate (see Tsai, Liao and Chiang, 2009). However, this result is different than the results in Figure 2b, showing that there is a negative duration for the mortgage contract. Our results are similar to the reports that maintain that a negative duration appears typically in severely default-risky fixed income securities (see Leland and Toft, 1996; Nakamura, 2001). Thus, assuming that the prepayment rate is a deterministic value and ignoring the effects of the interest rate on the prepayment rate will cause an incorrect inference about the mortgage duration. Using such an unreasonable inference to manage the interest rate risk of a mortgage could result in a large loss if the interest rate is greatly changed.

Figure 3b also shows that the mortgage durations differ greatly from one another in the three Lévy models. As shown in the figure, if the interest rate increases from 2% to 7%, the duration changes from -2.8773 to -18.2080 with the normal model. However, with the VG model, the duration changes only from -3.3760 to -8.5687. For the NIG model, the range of duration is from -3.4070 to -9.2097. Thus, the variance of the durations estimated by the normal model is larger than that with both the VG and NIG models. In other words, the normal model may overestimate the influence of the interest rate risk on the mortgage value, because duration in this model has higher variance than with the other two models. This could cause one to adopt an unsuitable hedging strategy. In contrast, from the viewpoint of risk management, the VG model is better for the valuation because the variance of the duration is the smallest among the three models. Accordingly, we conclude that the VG model is better for the valuation of a mortgage contract than the normal and NIG models, no matter whether the concern is on the fit of the state variables or on the management of the interest rate risk.

6. Conclusion

Lévy models have become increasingly popular and more apparent in the asset pricing literature. In this study, our main purpose was to use a Lévy model to develop a general mortgage pricing framework. Our model provides the following contributions. First, we specify that the state variables (i.e., the interest rate and the housing return rate) follow OU-Lévy processes. The OU process has the characteristic of mean reversion and its cumulative value is automatically restricted by a boundary value. Thus, the estimates for the prepayment and default values, specified by the affine functions of the state variables, are more reasonable than with the model in Liao et al. (2008). In addition, we use a time-changed Lévy model to describe the variance of the state variables. Such specification can accommodate the general features of the state variables, such as jumps, stochastic volatility, heavy tails, excess kurtosis and skewness.

Second, because we model the termination hazard rates and the recovery rate to be multivariate affine functions of the state variables, these three rates also follow the time-changed Lévy processes. The main advantage of such specifications is their capacity to generate the various possible features for the termination hazard rates and recovery rate curves. Accordingly, our model has more general specifications on the valuation of a mortgage than the traditional mortgage model, which uses the specification of affine jump diffusion (see Chiang and Tsai, 2012).

Finally, our study provides a closed-form formula for mortgage valuation based on the specifications that the termination hazard rates and the recovery rate follow time-changed Lévy processes. This formula can help fixed-income investors effectively manage the duration and complexity of mortgage portfolios and quickly determine diversification strategies.

Through the numerical results, we discussed mortgage values under the three popular Lévy models: normal, VG and NIG. Our results reveal that the VG model best fits the distributions of residuals for the interest rate and the housing return rate. Regarding interpretation of the VG model, the drift of the residuals for these two processes of the state variables can best be interpreted as Brownian motion, where time is changed by a gamma process. Thus, specifying the state variables by following a VG model should be superior to the normal process for assessing goodness-of-fit.

Market participants usually assume that the prepayment rate is a deterministic value (e.g., the PSA prepayment model) when valuating mortgage securities. We show that such a specification could lead to an incorrect pricing result and then cause a mistake in the calculation of the duration. Our numerical results show that if one ignores the influences of interest rate on the termination hazard rate, the calculated mortgage value is negatively correlated with the interest rate and the mortgage duration is a positive value. However, if the prepayment rate is strongly influenced by the interest rate, as in our numerical example, there should be a positive relationship between the mortgage value and the interest rate, and the mortgage duration should be a negative value.

Moreover, our results reveal that the mortgage value calculated using the normal model is smaller than that using the VG and NIG models. The normal model may overestimate the influence of the interest rate risk on the mortgage value because duration in this model has a higher variance than in the other two models. Thus, one could undertake an unsuitable hedging strategy when using the normal model.

We also infer that the VG model is better than the NIG model for valuating a mortgage. The first reason is that the VG model is better able to capture the processes of the interest rate and the housing return rate. The second reason is that the VG model is better at managing the interest rate risk because the variance of the duration is smaller than with the NIG model. Thus, we conclude that mortgage valuation using the VG model should be better than with using the normal or NIG models. In view of that, the VG model developed in this study more reasonably captures the actual situation and thus can improve both the accuracy of mortgage valuation and the efficiency of hedge strategy. Our model and findings should help market participants accurately value a mortgage and calculate its duration.

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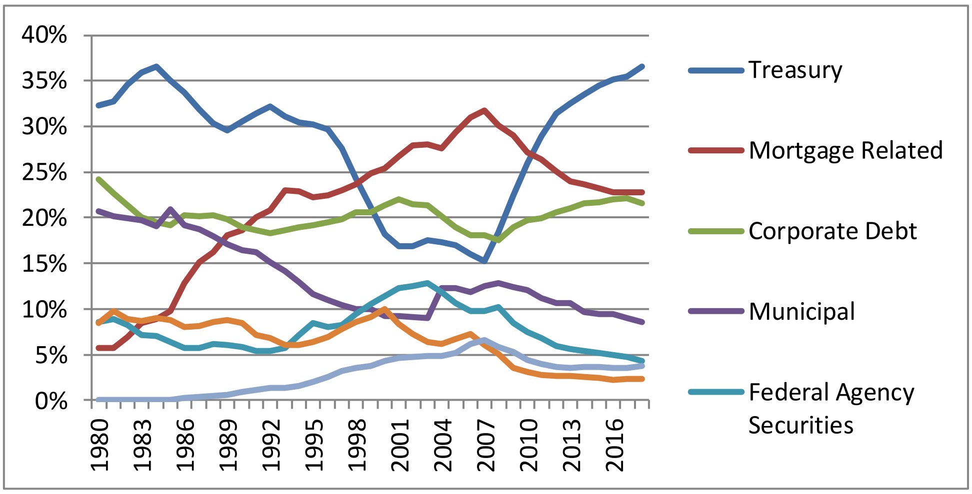
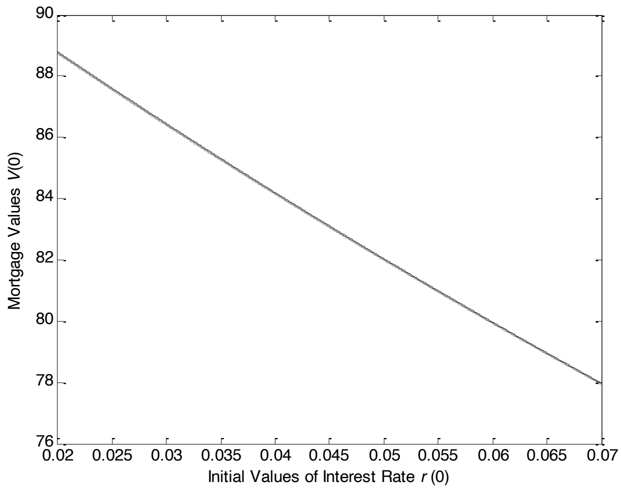
Figure 1: Share of outstanding for U.S. bond market from 1980-2018



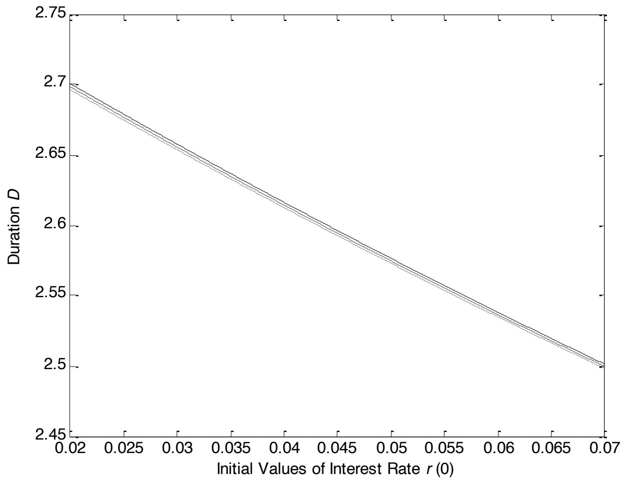
Source: SIFMA (Securities Industry and Financial Markets Association) Statistics
http://www.sifma.org/research/statistics.aspx

Figure 2: Sensitivity analyses of the influence of the initial values of the interest rate on mortgage value and duration when the termination rate and recovery rate are constant values

2a. Mortgage value corresponding to the different initial values of the interest rate



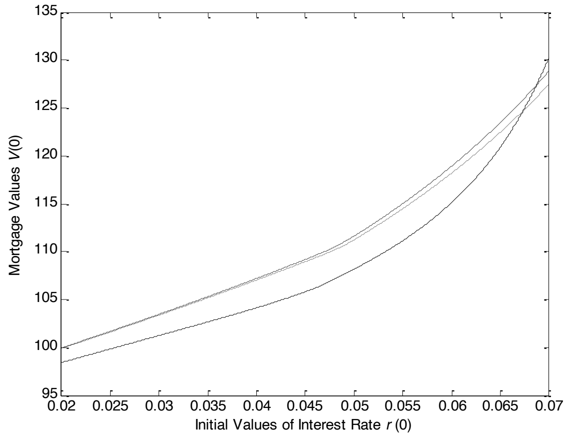
2b. Mortgage duration corresponding to the different initial values of the interest rate



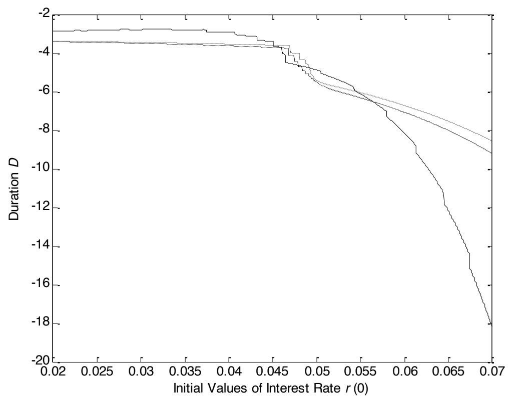
Note: The coefficients of the affine functions are $\varphi_0^{\theta} = 0.3812$, $\varphi_r^{\theta} = 0$, $\varphi_H^{\theta} = 0$, $\varphi_0^{\pi} = 0.0424$, $\varphi_r^{\pi} = 0$, $\varphi_H^{\pi} = 0$, $\varphi_0^{\rho} = 0.4017$, $\varphi_r^{\rho} = 0$ and $\varphi_H^{\rho} = 0$. The solid line represents the estimated values under the normal model, the dotted represents the estimated values under the VG model and the dashed line represents the estimated values under the NIG model. The figures show there is a negative relationship between the mortgage value when the initial value of the interest rate the termination rate and recovery rate are assumed to be constant values. Moreover, under these specifications, duration is a positive value and is negatively correlated with the initial value of the interest rate.

Figure 3: Influence of the interest rate on the mortgage value and its duration under the three Lévy models

3a. Mortgage value corresponding to the different initial values of the interest rate



3b. Mortgage duration corresponding to the different initial values of the interest rate



Note: The solid line represents the estimated values under the normal model, the dotted line represents the estimated values under the VG model and the dashed line represents the estimated values under the NIG model. The basic parameters are $M(0)=\$100$, $c=10\%$ and $r_H(0)=2\%$. We let $r(0)$ range from 0.02 to 0.07. The coefficients of the affine functions are φ_0^θ = 0.3812, φ_r^θ = -4.6498, φ_H^θ = 0.7846, φ_0^π = 0.0424, φ_r^π = -0.8846, φ_H^π = -0.0846, φ_0^ρ = 0.4017, φ_r^ρ = 3.7656 and φ_H^ρ = 1.3112. The parameter values of the three Lévy models for the interest rate and the housing return rate can be found in Table 3. Figure 3a shows that there is a positive relationship between the mortgage value and the initial value of the interest rate when the coefficient of prepayment (φ_r^θ) is a negative value. Figure 3b shows that duration has a negative value and is negatively correlated with the initial value of the interest rate when φ_r^θ is a negative value. When the initial value of the interest rate goes up, the values of the mortgage duration estimated by the three Lévy models have a wider disparity. In this situation, the mortgage duration estimated by the normal model is much less than for the VG and NIG models.

Table 1: Summary statistics for the state variables, termination rate and recovery rate

	Interest Rate	Housing Price Index	Prepayment Probability	Default Probability	Recovery Rate
Mean	0.0204	176.7300	0.2146	0.0248	0.4845
Standard Deviation	0.0164	32.1160	0.1295	0.0233	0.1617
Maximum	0.0502	226.2900	0.6085	0.0774	0.7649
Median	0.0167	168.7900	0.1628	0.0118	0.5084
Minimum	0.0003	122.8900	0.0574	0.0031	0.1125
Sample Number	110	110	110	110	110

Note: This table shows the means, standard deviations, medians, and maximum and minimum values for short-term interest rates, the housing price index, prepayment probabilities, default probabilities and recovery rates. Data for the last three variables are based on first-lien, 30-year fixed-rate mortgages taken from the CoreLogic LoanPerformance Securities Database. Housing prices were obtained from Standard and Poor's Case-Shiller 10-City Home Price Index. The short-term interest rate is for 3-month U.S. treasury bills. Our sample period from September 2001 to October 2010 yielded 110 observations for each variable.

Table 2: Estimates of the parameters for the drift term in the processes of the interest rate and the housing return rate

	θ_j^0	θ_j^1	a_j
Interest rate	2.9259×10^{-4} (0.55448)	-1.8492×10^{-6} (-0.81911)	-8.8384×10^{-3} (1.048)
Housing return rate	1.5243×10^{-3} *** (3.0384)	-5.1887×10^{-6} * (-1.6641)	0.34354*** (7.2316)

Note: the regression models are shown in Equations (38) and (39). The t-statistics appear in parentheses. θ_j^0, θ_j^1 and a_j for $j=r,H$ represent the parameter values in the processes of the interest rate and the housing return rate. ***significant at the 1% level. **significant at the 5% level. *significant at the 10% level.

Table 3: Estimates for the parameters of the interest rate and the housing return rate in the three Lévy models

Normal model	σ_N			$\max l_n(\xi)$
Interest rate	0.0053 (0.1577)			5.8391×10^{-4}
The housing return rate	0.0071*** (0.0000)			7.8589×10^{-4}
VG model	σ_{VG}	v_{VG}	θ_{VG}	
Interest rate	0.0054*** (0.0000)	257.3063*** (0.0000)	1.8292×10^{-6}*** (0.0000)	8.1119×10^{-4}
The housing return rate	0.0074*** (0.0000)	278.5796*** (0.0000)	3.1496×10^{-7} (0.4963)	7.8628×10^{-4}
NIG model	σ_{NIG}	v_{NIG}	θ_{NIG}	
Interest rate	6.3216×10^{-10} (0.4449)	37.2515*** (0.0000)	-0.8033*** (0.0000)	7.6876×10^{-6}
The housing return rate	9.5083×10^{-5} (0.4964)	0.9469*** (0.0000)	-0.0532*** (0.0000)	5.8972×10^{-4}

Note: This table shows the estimates of the parameters for the three types of processes of the interest rate and the housing return rate using the maximum empirical likelihood method; "normal model", "VG model" and "NIG model" mean that the Lévy processes are specified as the normal model, variance gamma model, and negative inverse Gaussian model, respectively. The definition for each Lévy model can be found in Section 3. $\max l_n(\hat{\xi})$ is the maximum empirical log-likelihood ratio given estimates $\hat{\xi}$. P-values appear in parentheses. ***significant at the 1% level. **significant at the 5% level. *significant at the 10% level.

Table 4: The calculated mortgage value under three Lévy models

Lévy model	**Survival Value**	**Prepayment Value**	**Default Value**	**Theoretical mortgage value**
Normal	31.8618	67.7755	1.5976	101.2349
VG	30.4477	70.6049	2.3038	103.3564
NIG	30.7979	70.3599	2.2927	103.4505

Note: This table shows the mortgage values under the three Lévy models. The basic parameter values adopted to operationalize our model are as follows: $M(0)=\$100$, $c=10\%$, $r(0)=3\%$ and $r_H(0)=2\%$. The coefficients of the affine functions are $\varphi_0^\theta = 0.3812$, $\varphi_r^\theta = -4.6498$, $\varphi_H^\theta = 0.7846$, $\varphi_0^\pi = 0.0424$, $\varphi_r^\pi = -0.8846$, $\varphi_H^\pi = -0.0846$, $\varphi_0^\rho = 0.4017$, $\varphi_r^\rho = 3.7656$ and $\varphi_H^\rho = 1.3112$. The parameters of each OU-Lévy process for the interest rate and the housing return rate can be found in Tables 2 and 3.

Appendix A:

This appendix shows how to obtain the formulas for $\Psi(t,u)$, $\theta^*(u)$, $\pi^*(u)$, $\rho^*(u)$ and $\Psi^{\pi,\rho}(t,u)$ in Equations (15), (16) and (17). The procedures can also be found in Tsai and Chiang (2012). According to Equations (9) and (12), we obtain

$$\begin{aligned}
&A_r\Xi_r(t,T) + A_H\Xi_H(t,T) \\
&\quad = A_r\Xi_r^{\mu}(t,T) + \int_t^T A_r a_r^{-1}(1-e^{-a_r(v-t)})dL_r(v) \\
&\qquad + A_H\Xi_H^{\mu}(t,T) + \int_t^T A_H a_H^{-1}(1-e^{-a_H(v-t)})\rho_{r,H}dL_r(v) + \int_t^T A_H a_H^{-1}(1-e^{-a_H(v-t)})dL_H(v) \\
&\quad = A_r\Xi_r^{\mu}(t,T) + \int_t^T (A_r\eta_r(v) + A_H\eta_H(v)\rho_{r,H})dL_r(v) \\
&\qquad + A_H\Xi_H^{\mu}(t,T) + \int_t^T A_H\eta_H(v)dL_H(v). \qquad \text{(A1)}
\end{aligned}$$

Thus, when the MGF of $-A_I(t,T) - A'\Xi(t,T)$ is expressed as $\Psi(t,T)$, we have the following:

$$\begin{aligned}
\Psi(t,u) &\equiv E[\exp(-A_I(t,u) - A'\Xi(t,u))] \\
&= E[\exp(-A_I(t,u) - A_r\Xi_r(t,u) - A_H\Xi_H(t,u))] \\
&= \exp(-A_I(t,u) + \overline{\Xi}_r(-A_r,t,u) + \kappa_r^{\Xi}(-A_r,-A_H,t,u) \\
&\qquad + \overline{\Xi}_H(-A_H,t,u) + \kappa_H^{\Xi}(-A_H,t,u)). \qquad \text{(A2)}
\end{aligned}$$

This is Equation (15).

The MGF of $B'Z(u)$ can be shown as follows:

$$\begin{aligned}
M_\theta(c_1,c_2,t,u) &= E[\exp((-c_1A_I(t,u) + c_2\varphi_0^{\theta}(u)) + (-c_1A' + c_2A_\theta' I_{(s=u)})\Xi(t,u))] \\
&= \exp((-c_1A_I(t,u) + c_2\varphi_0^{\theta}(u)) \\
&\qquad + \overline{\Xi}_r(-c_1A_r + c_2\varphi_r^{\theta}I_{(s=u)},t,u) + \kappa_r^{\Xi}(-c_1A_r + c_2\varphi_r^{\theta}I_{(s=u)},-c_1A_H + c_2\varphi_H^{\theta}I_{(s=u)},t,u) \\
&\qquad + \overline{\Xi}_H(-c_1A_H + c_2\varphi_H^{\theta}I_{(s=u)},t,u) + \kappa_H^{\Xi}(-c_1A_H + c_2\varphi_H^{\theta}I_{(s=u)},t,u))). \qquad \text{(A3)}
\end{aligned}$$

In Equation (A3), we have

$$\overline{\Xi}_r(-c_1A_r + c_2\varphi_r^\theta I_{(s=u)}, t, u)$$

$$= \int_t^u (-c_1A_r + c_2\varphi_r^\theta I_{(s=u)})(r(t)e^{-a_r(s-t)} + \int_t^s e^{-a_r(s-v)}\theta(v)dv)ds\,. \tag{A4}$$

Therefore, we can obtain

$$\frac{\partial\overline{\Xi}_r(-c_1A_r + c_2\varphi_r^\theta I_{(s=u)}, t, u)}{\partial c_2}\Big|_{c_1=1,c_2=0}$$

$$= \int_t^u \varphi_r^\theta I_{(s=u)}(r(t)e^{-a_r(s-t)} + \int_t^s e^{-a_r(s-v)}\theta(v)dv)ds$$

$$= \varphi_r^\theta(r(t)e^{-a_r(u-t)} + \int_t^u e^{-a_r(u-v)}\theta(v)dv) = \varphi_r^\theta r^\mu(u)\,. \tag{A5}$$

Moreover, we have

$$\frac{\partial\overline{\Xi}_H(-c_1A_H + c_2\varphi_H^\theta I_{(s=u)}, t, u)}{\partial c_2}\Big|_{c_1=1,c_2=0}$$

$$= \varphi_H^\theta(r_H(t)e^{-a_H(u-t)} + \int_t^u e^{-a_H(u-v)}\theta_H(v)dv) = \varphi_H^\theta r_H^\mu(u)\,. \tag{A6}$$

Also, we have

$$\frac{\partial\kappa_r^\Xi(-c_1A_r + c_2\varphi_r^\theta I_{(s=u)}, -c_1A_H + c_2\varphi_H^\theta I_{(s=u)}, t, u)}{\partial c_2} = \int_t^u \frac{\partial\kappa_r(b)}{\partial b}\frac{\partial b}{\partial c_2}\ )ds\,, \tag{A7}$$

where $b = -c_1(A_r\eta_r(s) + A_H\eta_H(s)\rho_{r,H}) + c_2(\varphi_r^\theta I_{(s=u)}\eta_r(s) + \varphi_H^\theta I_{(s=u)}\eta_H(s)\rho_{r,H})$.

Thus, we get

$$\frac{\partial\kappa_r^\Xi(-c_1A_r + c_2\varphi_r^\theta I_{(s=u)}, -c_1A_H + c_2\varphi_H^\theta I_{(s=u)}, t, u)}{\partial c_2}\Big|_{c_1=1,c_2=0}$$

$$= \int_t^u \frac{\partial\kappa_r(b)}{\partial b}(\varphi_r^\theta I_{(s=u)}\eta_r(s) + \varphi_H^\theta I_{(s=u)}\eta_H(s)\rho_{r,H})\ )ds\,\Big|_{c_1=1,c_2=0}$$

$$= (\varphi_r^\theta\eta_r(u) + \varphi_H^\theta\eta_H(u)\rho_{r,H})\kappa_r'(-A_r\eta_r(u) - A_H\rho_{r,H}\eta_H(u))\,. \tag{A8}$$

Likewise, we have

$$\frac{\partial\kappa_H^\Xi(-c_1A_H + c_2\varphi_H^\theta I_{(s=u)}, t, u)))}{\partial c_2}\Big|_{c_1=1,c_2=0} = \varphi_H^\theta\eta_H(u)\kappa_H'(A_H\eta_H(u))\,. \tag{A9}$$

According to Equations (A3)–(A9), we obtain the following result:

$$\Psi^{P}(t,u)=\frac{\partial M_{\theta}(c_1,c_2,t,u)}{\partial c_2}\Big|_{c_1=1,c_2=0}$$

$$=\theta^{*}(u)\times\Psi(t,u), \tag{A10}$$

where

$$\theta^{*}(u)=\varphi_0^{\theta}+\varphi_r^{\theta}r^{\mu}(u)+\varphi_H^{\theta}r_H^{\mu}(u)$$

$$+(\varphi_r^{\theta}\eta_r(u)+\varphi_H^{\theta}\eta_H(u)\rho_{r,H})\kappa_r'(-A_r\eta_r(u)-A_H\rho_{r,H}\eta_H(u))$$

$$+\varphi_H^{\theta}\eta_H(u)\kappa_H'(A_H\eta_H(u)).$$

This is Equation (16).

The MGF of $D'_{\pi,\rho}Y(u)$ can be shown as follows:

$$M_{\pi,\rho}(d_1,d_2,d_3,t,u)=E[\exp(-d_1A_I(t,u)+d_2\varphi_0^{\pi}(u)+d_3\varphi_0^{\rho}(u)+(-d_1A'+d_2A_{\pi}'I_{(s=u)}+d_3A_{\rho}'I_{(s=u)})\Xi(t,u))]$$

$$=\exp((-d_1A_I(t,u)+d_2\varphi_0^{\pi}(u)+d_3\varphi_0^{\rho}(u))$$

$$+\overline{\Xi}_r(-d_1A_r+d_2\varphi_r^{\pi}I_{(s=u)}+d_3\varphi_r^{\rho}I_{(s=u)},t,u)$$

$$+\kappa_r^{\Xi}(-d_1A_r+d_2\varphi_r^{\pi}I_{(s=u)}+d_3\varphi_r^{\rho}I_{(s=u)},-d_1A_H+d_2\varphi_H^{\pi}I_{(s=u)}+d_3\varphi_r^{\rho}I_{(s=u)},t,u)$$

$$+\overline{\Xi}_H(-d_1A_H+d_2\varphi_H^{\pi}I_{(s=u)}+d_3\varphi_r^{\rho}I_{(s=u)},t,u)$$

$$+\kappa_H^{\Xi}(-d_1A_H+d_2\varphi_H^{\pi}I_{(s=u)}+d_3\varphi_r^{\rho}I_{(s=u)},t,u))). \tag{A11}$$

Accordingly, we have

$$\frac{\partial^2 M_{\pi,\rho}(d_1,d_2,d_3,t,u)}{\partial d_2\partial d_3}$$

$$=M_{\pi,\rho}(d_1,d_2,d_3,t,u)\Big(\frac{\partial\ln(M_{\pi,\rho}(d_1,d_2,d_3,t,u))}{\partial d_2}\times\frac{\partial\ln(M_{\pi,\rho}(d_1,d_2,d_3,t,u))}{\partial d_3}$$

$$+\frac{\partial\ln(M_{\pi,\rho}(d_1,d_2,d_3,t,u))}{\partial d_2\partial d_3}\Big). \tag{A12}$$

We let

$$\pi^*(u) = \frac{\partial \ln(M_{\pi,\rho}(d_1, d_2, d_3, t, u))}{\partial d_2} |_{d_1=1,d_2=0,d_3=0}$$

$$= \varphi_0^\pi + \frac{\partial \overline{\Xi}_r(\omega_r, t, u)}{\partial d_2} + \frac{\partial \kappa_r^\Xi(\omega_r, \omega_H, t, u)}{\partial d_2}$$

$$+ \frac{\partial \overline{\Xi}_H(\omega_H, t, u)}{\partial d_2} + \frac{\partial \kappa_H^\Xi(\omega_H, t, u)}{\partial d_2} |_{d_1=1,d_2=0,d_3=0}; \quad \text{(A13)}$$

$$\rho^*(u) = \frac{\partial \ln(M_{\pi,\rho}(d_1, d_2, d_3, t, u))}{\partial d_3} |_{d_1=1,d_2=0,d_3=0}$$

$$= \varphi_0^\rho + \frac{\partial \overline{\Xi}_r(\omega_r, t, u)}{\partial d_3} + \frac{\partial \kappa_r^\Xi(\omega_r, \omega_H, t, u)}{\partial d_3}$$

$$+ \frac{\partial \overline{\Xi}_H(\omega_H, t, u)}{\partial d_3} + \frac{\partial \kappa_H^\Xi(\omega_H, t, u)}{\partial d_3} |_{d_1=1,d_2=0,d_3=0}; \text{ and} \quad \text{(A14)}$$

$$\Psi^{\pi,\rho}(t,u) = \frac{\partial \ln(M_{\pi,\rho}(d_1, d_2, d_3, t, u))}{\partial d_2 \partial d_3} |_{d_1=1,d_2=0,d_3=0}$$

$$= \frac{\partial^2 \overline{\Xi}_r(\omega_r, t, u)}{\partial d_2 \partial d_3} + \frac{\partial^2 \kappa_r^\Xi(\omega_r, \omega_H, t, u)}{\partial d_2 \partial d_3}$$

$$+ \frac{\partial^2 \overline{\Xi}_H(\omega_H, t, u)}{\partial d_2 \partial d_3} + \frac{\partial^2 \kappa_H^\Xi(\omega_H, t, u)}{\partial d_2 \partial d_3} |_{d_1=1,d_2=0,d_3=0}, \quad \text{(A15)}$$

where $\omega_r = -d_1 A_r + d_2 \varphi_r^\pi I_{(s=u)} + d_3 \varphi_r^\rho I_{(s=u)}$, and $\omega_H = -d_1 A_r + d_2 \varphi_r^\pi I_{(s=u)} + d_3 \varphi_r^\rho I_{(s=u)}$.

According to the method shown in Equations (A5)–(A9), one can obtain the formulas $\frac{\partial \overline{\Xi}_r(\omega_r, t, u)}{\partial m}$, $\frac{\alpha \kappa_r^\Xi(\omega_r, \omega_H, t, u)}{\partial m}$, $\frac{\partial \overline{\Xi}_H(\omega_H, t, u)}{\partial m}$ and $\frac{\partial \kappa_H^\Xi(\omega_H, t, u)))}{\partial m}$ for $m = d_2, d_3$. According to these results, we obtain the following formulas for $\pi^*(u)$ and $\rho^*(u)$:

$$\pi^*(u) = \varphi_0^\pi + \varphi_r^\pi r^\mu(u) + \varphi_H^\pi r_H^\mu(u)$$

$$+(\varphi_r^{\pi}\eta_r(u)+\varphi_H^{\pi}\eta_H(u)\rho_{r,H})\kappa_r'(-A_r\eta_r(u)-A_H\eta_H(u)\rho_{r,H})$$

$$+\varphi_H^{\pi}\eta_H(u)\kappa_H'(-A_H\eta_H(u))\text{, and}$$

$$\rho^*(u)=\varphi_0^{\rho}+\varphi_r^{\rho}r^{\mu}(u)+\varphi_H^{\rho}r_H^{\mu}(u)$$

$$+(\varphi_r^{\rho}\eta_r(u)+\varphi_H^{\rho}\eta_H(u)\rho_{r,H})\kappa_r'(-A_r\eta_r(u)-A_H\eta_H(u)\rho_{r,H})$$

$$+\varphi_H^{\rho}\eta_H(u)\kappa_H'(-A_H\eta_H(u)).$$

In addition, we have the following results:

$$\frac{\partial^2\Xi_i(\omega_i,t,u)}{\partial d_2\partial d_3}|_{d_1=1,d_2=0,d_3=0}=0\text{, for } i=r,H. \quad \text{(A16)}$$

$$\frac{\partial^2\kappa_r^{\Xi}(\omega_r,\omega_H,t,u)}{\partial d_2\partial d_3}|_{d_1=1,d_2=0,d_3=0}$$

$$=\int_t^u\left(\frac{\partial\kappa_r^2(b)}{\partial b^2}\frac{\partial(\omega_r+\omega_H)}{\partial d_2}\frac{\partial(\omega_r+\omega_H)}{\partial d_3}+\frac{\partial\kappa_r(b)}{\partial b}\frac{\partial^2(\omega_r+\omega_H)}{\partial d_2\partial d_3}\right)ds\,|_{d_1=1,d_2=0,d_3=0}$$

$$=\int_t^u\frac{\partial\kappa_r^2(b)}{\partial b^2}((\varphi_r^{\pi}I_{(s=u)}\eta_r(s)+\varphi_H^{\pi}I_{(s=u)}\eta_H(s)\rho_{r,H})$$

$$\times(\varphi_r^{\rho}I_{(s=u)}\eta_r(s)+\varphi_H^{\rho}I_{(s=u)}\eta_H(s)\rho_{r,H})\,)ds\,|_{d_1=1,d_2=0,d_3=0}$$

$$=(\varphi_r^{\pi}\eta_r(u)+\varphi_H^{\pi}\eta_H(u)\rho_{r,H})(\varphi_r^{\rho}\eta_r(u)+\varphi_H^{\rho}\eta_H(u)\rho_{r,H}))$$

$$\times\kappa_r''(-A_r\eta_r(s)-A_H\rho_{r,H}\eta_H(s)). \quad \text{(A14)}$$

Also, we have

$$\frac{\partial\kappa_H^{\Xi}(-d_1A_H+d_2\varphi_H^{\pi}I_{(s=u)}+d_3\varphi_r^{\rho}I_{(s=u)},t,u))}{\partial d_2\partial d_3}|_{d_1=1,d_2=0,d_3=0}=\varphi_H^{\pi}\eta_H(u)\varphi_H^{\rho}\eta_H(u)\kappa_H''(A_H\eta_H(u)). \quad \text{(A15)}$$

Thus, we can obtain the following result:

$$\frac{\partial^2M_{\pi,\rho}(d_1,d_2,d_3,t,u)}{\partial d_2\partial d_3}|_{d_1=1,d_2=0,d_3=0}=(\pi^*(u)\rho^*(u)+\Psi^{\pi,\rho}(t,u))\times\Psi(t,u). \quad \text{(A16)}$$

This is Equation (17).

Appendix B:

This appendix shows how to obtain the maximum empirical likelihood (Qin and Lawless, 1994; Elgin, 2011). We let $\varepsilon(t)$ ($\varepsilon(t)=\varepsilon_r(t)$ or $\varepsilon_H(t)$), $t=1,\cdots,n$ be the observations of the Lévy process. The empirical characteristic function is defined as:

$\phi_n(m)=\frac{1}{n}\sum_{t=1}^{n}e^{im\varepsilon(t)}$, where $i=\sqrt{-1}$.

We let $\phi(m,\xi)$ be the theoretical characteristic function, given parameters ξ (e.g., $\xi=[\sigma_N]$ for the normal process, $\xi=[\sigma_{VG},\nu_{VG},\theta_{VG}]$ for the VG process and $\xi=[\sigma_{NIG},\nu_{NIG},\theta_{NIG}]$ for the NIG process). We express

$$\phi(m,\xi)=\phi^R(m,\xi)+i\phi^I(m,\xi),$$

where $\phi^R(m,\xi)$ is the real part of $\phi(m,\xi)$ and $\phi^I(m,\xi)$ is the imaginary part of $\phi(d,\xi)$.

We then define

$$h(m,\varepsilon(t),\xi)=e^{im\varepsilon(t)}-\phi(m,\xi).$$

Because $e^{im\varepsilon(t)}=\cos(m\varepsilon(t))+\sin(m\varepsilon(t))$, we have

$$h(m,\varepsilon(t),\xi)\quad=\cos(m\varepsilon(t))-\phi^R(m,\xi)+i(\sin(m\varepsilon(t))-\phi^I(m,\xi)).$$

Using a grid method, we let m be divided into k points, that is, $m\equiv[m_1\quad\cdots\quad m_k]$. Because $E[\phi_n(m)]=\phi(m\,|\,\xi)$, we have $E[h(m,\varepsilon(t),\xi)]=0$. Accordingly, we have the following condition when the estimates are obtained by the empirical and theoretical characteristic functions:

$$\sum_{t=1}^{n} p_t h(\varepsilon(t),\xi) = 0, \tag{C1}$$

where

p_t is the probability of $\varepsilon(t)$;

$h(\varepsilon(t),\xi) = [h^R(\varepsilon(t),\xi) \quad h^I(\varepsilon(t),\xi)]'$, a vector with $2k \times 1$ ranks;

$h^R(\varepsilon(t),\xi) = [\cos(m_1\varepsilon(t)) - \phi^R(m_1,\xi) \quad \cdots \quad \cos(m_k\varepsilon(t)) - \phi^R(m_k,\xi)]$, a vector with $k \times 1$ ranks; and

$h^I(\varepsilon(t),\xi) = [\sin(m_1\varepsilon(t)) - \phi^I(m_1,\xi) \quad \cdots \quad \sin(m_k\varepsilon(t)) - \phi^I(m_k,\xi)]$, a vector with $k \times 1$ ranks.

Thus, using the Lagrange multiplier method, the Lagrange system is expressed as follows:

$$\max_{\xi} l_n(\xi) = -\sum_{t=1}^{n} \log(np_t),$$

$$s.t.\ p_t \geq 0,\ \sum_{t=1}^{n} p_t = 1,\ \sum_{t=1}^{n} p_t h(\varepsilon(t),\xi) = 0,$$

where $l_n(\xi)$ is the empirical log-likelihood ratio function. The former condition is expressed as the properties of the probability and the latest condition is given by Equation (C1).

Solving this system (for the details, see Qin and Lawless, 1994 and Elgin, 2011), the maximum empirical likelihood estimator is obtained by maximizing

$$l_n(\xi) = -\sum_{t=1}^{n} \log(1 + \eta'(\xi) h(\varepsilon(t),\xi)),$$

where $\eta(\xi)$, a vector with $2k \times 1$ ranks, is the Lagrange multiplier and can be solved by the following equation:

$$\frac{1}{n}\sum_{t=1}^{n}\frac{1}{1+\eta'(\xi)h(\varepsilon(t),\xi)}h(\varepsilon(t),\xi)=0\,.$$

We then obtain the estimator, expressed as: $\hat{\xi}=\arg\max l_n(\xi)$.

Appendix C:

This appendix shows the moment method for the estimation. For a process $\varepsilon(t)$ (i.e., $\varepsilon(t)=\varepsilon_r(t)$ or $\varepsilon_H(t)$), $t=1,\cdots,n$, the sample mean (denoted as $\bar{\varepsilon}$), sample variance (denoted as s_ε^2), and sample skewness (denoted as k_ε) can be expressed as

$$\bar{\varepsilon}=\frac{\sum_{t=1}^{n}\varepsilon(t)}{n},\quad s_\varepsilon^2=\frac{\sum_{t=1}^{n}(\varepsilon(t)-\bar{\varepsilon})^2}{n},\text{ and }\ k_\varepsilon=\frac{\frac{1}{n}\sum_{t=1}^{n}(\varepsilon(t)-\bar{\varepsilon})^3}{(\frac{1}{n}\sum_{t=1}^{n}(\varepsilon(t)-\bar{\varepsilon})^2)^{\frac{3}{2}}}.$$

We denote the population mean, population variance and population skewness as μ_ε, σ_ε^2 and ζ_ε respectively. In the following, we present μ_ε, σ_ε^2 and ζ_ε for each Lévy model. For the normal model, we have $\sigma_\varepsilon^2=\sigma_N^2$. For the VG model, the mean, variance and skewness can be expressed as

$$\mu_\varepsilon=\theta_{VG},\quad \sigma_\varepsilon^2=\sigma_{VG}^2+\nu_{VG}\theta_{VG}^2,\text{ and }\ \zeta_\varepsilon=\frac{\theta_{VG}\nu_{VG}(3\sigma_{VG}^2+2\nu_{VG}\theta_{VG}^2)}{(\sigma_{VG}^2+\nu_{VG}\theta_{VG}^2)^{\frac{3}{2}}}.$$

For the NIG model, the mean, variance and skewness can be expressed as

$$\mu_\varepsilon=\frac{\sigma_{NIG}\theta_{NIG}}{\sqrt{\nu_{NIG}^2-\theta_{NIG}^2}},\quad \sigma_\varepsilon^2=\nu_{NIG}^2\sigma_{NIG}(\nu_{NIG}^2-\theta_{NIG}^2)^{-\frac{3}{2}};\text{ and}$$

$$\zeta_\varepsilon=3\theta_{NIG}\nu_{NIG}^{-1}\sigma_{NIG}^{-\frac{1}{2}}(\nu_{NIG}^2-\theta_{NIG}^2)^{-\frac{1}{4}}.$$

Using the sample moment to estimate the parameters, we have $\sigma_\varepsilon^2=s_\varepsilon^2$ for the normal model. For the VG and NIG models, we use three sample statistic values to obtain the model's parameters. The functions are expressed as follows:

$$\mu_\varepsilon=\bar{\varepsilon},\quad \sigma_\varepsilon^2=s_\varepsilon^2,\text{ and }\ \zeta_\varepsilon=k_\varepsilon.$$

Using the above method, we obtain the estimates of the parameters for the interest rate and the housing return rate in the Lévy models. Under the normal model, we have σ_N = 7.6684×10^{-3} for the interest rate and σ_N = 8.0534×10^{-3} for the housing return rate. Under the VG model, we have σ_{VG} = 7.6398×10^{-3}, ν_{VG} = 136.1238 and θ_{VG} = -2.0914×10^{-5} for the interest rate; and σ_{VG} = 7.3352×10^{-3}, ν_{VG} = 157.8497 and θ_{VG} = -6.5050×10^{-6} for the housing return rate. Under the NIG model, we have σ_{NIG} = 6.8279×10^{-4}, ν_{NIG} = 20.4117 and θ_{NIG} = -0.8968 for the interest rate; and σ_{NIG} = 6.5399×10^{-5}, ν_{NIG} = 19.9852 and θ_{NIG} = -0.1011 for the housing return rate. Then, we use these values as the initial values when applying the maximum empirical likelihood method, as shown in Appendix B.